ORTHOGONAL POLYNOMIALS AND GENERALIZED OSCILLATOR ALGEBRAS

V.V. Borzov

Department of Mathematics, St.Petersburg University of Telecommunications, 191065, Moika 61, St.Petersburg, Russia

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For any orthogonal polynomials system on real line we construct an appropriate oscillator algebra such that the polynomials make up the eigenfunctions system of the oscillator hamiltonian. The general scheme is divided into two types: a symmetric scheme and a non-symmetric scheme. The general approach is illustrated by the examples of the classical orthogonal polynomials: Hermite, Jacobi and Laguerre polynomials. For these polynomials we obtain the explicit form of the hamiltonians, the energy levels and the explicit form of the impulse operators.

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1. Introduction

The connection of orthogonal polynomials with the classical groups ([1]) as well as with the quantum ones ([2]) is well known. We discuss here the connection of orthogonal polynomials with the Heisenberg algebra of generalized (deformed ([3, 4, 5]) as an example) oscillator. Recall that (see, for example, [6]) the Hermite polynomials (after multiplication by $exp(-x^2)$) make up the eigenfunctions system of the energy operator for the quantum mechanical harmonic oscillator. Many of the known q-Hermite polynomials ([7, 8, 9]) are also the eigenfunctions of the energy operator for a deformed oscillator. It is well known that orthogonal polynomials ,which in a sense generalize the Hermite polynomials , appear in the analysis of the irreducible representations of the algebra of an appropriate oscillator. In this paper we propose another way of looking at the connection of orthogonal polynomials with some generalized oscillator algebras. Namely, given an orthogonal polynomials system, we construct an appropriate oscillator algebra so that the polynomials make up a eigenfunctions system of the oscillator hamiltonian.

The aim of this paper is to present the classical orthogonal polynomials as eigenfunctions of an energy operator for a generalized oscillator. Let us take a brief look at the considered approach. A preassigned Hilbert space with an orthogonal polynomials systems (for instance, one of the above-mentioned classical polynomials systems) as a basis is considered as a Fock space. As it usually is, we define the ladder operators (annihilation) a^- and (creation) a^+ as well as the number operator N in this space. By a standard manner we use these operators to build up the following selfadjoint operators: the position operator X, the momentum operator P as well as the energy operator (hamiltonian) $H = X^2 + P^2$. By analogy with the usual Heisenberg algebra these operators generate an algebra, which naturally is called a generalized oscillator algebra. It turns out that the operator H has a simple discrete spectrum. The initial orthogonal polynomials set is an eigenfunctions system of the energy operator H. Via the Poisson kernel of this system is determined a generalized Fourier transform, which establishes the usual link between the operators X and P. The energy operator H is invariable under the action of this transform. The

explicit form of the Poisson kernels for the classical orthogonal polynomials (the analog of the Mehler formula [10]) see in [11], [12], [13]. The orthogonal polynomials systems (OPS) can be further divided into two types: symmetric systems and non-symmetric systems. OPS is called a symmetric system if the orthogonality measure for these polynomials is symmetric about the origin; otherwise it is called a non-symmetric system. In the former case the Jacobi matrix of the operator X (in the Fock representation) has the trivial diagonal. Note that the above-mentioned oscillator algebra arise only in the first case. In the latter case one can also construct a generalized oscillator algebra. However the oscillator hamiltonian takes the standard form only in new "coordinate-impulse" operators , which can result from the previous operators X and P by a rotation.

2. Symmetric scheme

2.1. Let μ be a positive Borel measure on the real line R^1 such that

$$\int_{-\infty}^{\infty} \mu(dx) = 1, \qquad \mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1}\mu(dx) = 0, \qquad k = 0, 1, \dots . \tag{2.1}$$

The measure μ is called a symmetric probability measure. By $\mathtt{H}$ we denote the Hilbert space $L^2(R^1; \mu)$. Let $\{b_n\}_{n=0}^{\infty}$, $\quad b_n > 0, \quad n = 0, 1, \dots$ be a positive sequence defined by the algebraic equations system

$$\mu_{2k} = b_0^2 \cdot (b_0^2 + b_1^2) \cdots \sum_{j=0}^{k-1} b_j^2, \qquad k = 0, 1, \dots , \tag{2.2}$$

where

$$\mu_0 = 1, \qquad \mu_{2k} = \int_{-\infty}^{\infty} x^{2k}\mu(dx), \qquad k = 0, 1, \dots , \tag{2.3}$$

Obviously, there is the unique solution to the system (2.2)

$$b_0^2 = \mu_2, \qquad b_1^2 = \frac{\mu_4}{\mu_2} - \mu_2, \dots . \tag{2.4}$$

Definition 2.1. A polynomial set $\{\psi_n(x)\}_{n=0}^{\infty}$ is called a canonical polynomial system if it is defined by the following recurrence relations:

$$x\psi_n(x) = b_n\psi_{n+1}(x) + b_{n-1}\psi_{n-1}(x), \qquad n \geq 0, \qquad b_{-1} = 0, \tag{2.5}$$

$$\psi_0(x) = 1, \tag{2.6}$$

where the positive sequence $\{b_n\}_{n=0}^{\infty}$ is given.

Remark 2.2. 1. The canonical polynomial system $\{\psi_n(x)\}_{n=0}^{\infty}$ is uniquely determined by the symmetric probability measure μ.

2.The recurrence relations (2.5) give us the symmetric Jacobi matrix

$$J = \{b_{ij}\}_{i,j=0}^{\infty}$$

which has the positive elements $b_{i,i+1} = b_{i+1,i}, \quad i = 0, 1, \dots$ only distinct from zero. If the moment problem ([17]) for the matrix J is a determined one, then the canonical polynomial system $\{\psi_n(x)\}_{n=0}^{\infty}$ is completed in the space $\mathtt{H}$. Otherwise (when the moment problem is a undetermined one) the canonical polynomial system $\{\psi_n(x)\}_{n=0}^{\infty}$ is completed in the space $\mathtt{H}$ if and only if the measure μ is a N- extremal solution ([17]) of the moment problem for the matrix J.

The following theorem is true.

Theorem 2.3. *Let $\{\psi_n(x)\}_{n=0}^{\infty}$ be a set of real polynomials satisfying recurrence relations (2.5) and a initial condition (2.6); let μ be a symmetric probability measure on the real line R^1, that is the conditions (2.1) for μ are valid. The set $\{\psi_n(x)\}_{n=0}^{\infty}$ is a system of polynomials orthonormal with respect to the measure μ if and only if a positive sequence $\{b_n\}_{n=0}^{\infty}$ involved in the recurrence relations (2.5) is a solution of the system (2.2), where μ_{2k} are defined by (2.3).*

2.2. Let $\psi(x)$ be a real-valued function such that $\frac{1}{\psi(x)}$ is measurable with respect to the above-mentioned measure μ. Let us introduce new measure ν realised by

$$\nu(dx) = |\psi(x)|^{-2}\mu(dx). \tag{2.7}$$

Remark 2.4. 1. The function $|\psi(x)|^{-2}$ is locally integrable but it is not necessarily that $|\psi(x)|^{-2} \in L^1(R^1; \mu(dx)$.

2.In general, the conditions (2.1) break down for the measure ν.

Now we consider another Hilbert space $\mathtt{G} = L^2(R^1; \nu(dx))$ with the measure ν defined by (2.7). We define the functions system $\{\phi_n(x)\}_{n=0}^{\infty}$, $\phi_n(x) \in \mathtt{G}, n = 0, 1, \dots$ by

$$\phi_n(x) = \psi(x)\psi_n(x), \qquad n = 0, 1, \dots, \tag{2.8}$$

where the set $\{\psi_n(x)\}_{n=0}^{\infty}$ is a canonical polynomial system in above space $\mathtt{H}$.

The following statement is a simple consequence of the theorem 2.3.

Corollary 2.5. *If the system $\{\psi_n(x)\}_{n=0}^{\infty}$ is a canonical system of polynomials orthonormal with respect to the measure μ in the space $\mathtt{H}$, then the set $\{\phi_n(x)\}_{n=0}^{\infty}$,$\phi_n(x) \in \mathtt{G}, n = 0, 1, \dots$ defined by (2.8) is a orthonormal system in the space $\mathtt{G} = L^2(R^1; \nu(dx))$. Besides, this system satisfies the same recurrence relations (2.5) and the initial condition*

$$\phi_0(x) = \psi(x). \tag{2.9}$$

Remark 2.6. It is evident, that a completeness of system $\{\phi_n(x)\}_{n=0}^{\infty}$ in the space $\mathtt{G}$ is equivalent to the one of $\{\psi_n(x)\}_{n=0}^{\infty}$ in the space $\mathtt{H}$.

2.3. **The Poisson kernel.** For the reader's convenience we remind the definition of the Poisson kernel in the Hilbert space $\mathtt{F} = L^2(R^1; \rho(dx))$, where ρ is a positive Borel measure on the real line R^1. Let a set $\{\varphi_n(x)\}_{n=0}^{\infty}$ be an orthonormal basis in the space $\mathtt{F}$. From now on we will use the notation $\mathtt{F}$ instead of $\mathtt{G}$ or $\mathtt{H}$ if both spaces are regarded together. Let us denote by $\mathtt{F}_1$, $\mathtt{F}_2$ the first and second copies of the space $\mathtt{F}$ respectively:

$$\mathtt{F}_1 = L^2(R^1; \rho(dx)), \qquad \mathtt{F}_2 = L^2(R^1; \rho(dy)). \tag{2.10}$$

The Poisson kernel $\mathfrak{K}_{\mathtt{F}}(x, y; t)$ on $\mathtt{F}_1 \otimes \mathtt{F}_2$ is defined by the formula

$$\mathfrak{K}_{\mathtt{F}}(x, y; t) = \sum_{n=0}^{\infty} t^n \cdot \varphi_n(x) \cdot \varphi_n(y). \tag{2.11}$$

From (2.8) and (2.11) it follows that:

$$\mathfrak{K}_{\mathtt{G}}(x, y; t) = \psi(x)\psi(y)\mathfrak{K}_{\mathtt{H}}(x, y; t). \tag{2.12}$$

We define the integral operators $K_{\mathtt{F}}$: $\mathtt{F}_1 \longmapsto \mathtt{F}_2$ and $K'_{\mathtt{F}} : \mathtt{F}_2 \longmapsto \mathtt{F}_1$ by the following formulas

$$(K_{\mathtt{F}} f)(y) = \int_{-\infty}^{\infty} f(x)\mathfrak{K}_{\mathtt{F}}(x, y; t)\rho(dx), \tag{2.13}$$

$$(K'_{\mathtt{F}}g)(x) = \int_{-\infty}^{\infty} g(y)\overline{\mathfrak{K}_{\mathtt{F}}(x,y;t)}\rho(dy). \tag{2.14}$$

It is easy to prove the following lemmas.

Lemma 2.7. *Let the set $\{\varphi_n(x)\}_{n=0}^{\infty}$ be an orthonormal system in the space $\mathtt{F}_1$. If this system is completed in $\mathtt{F}_1$ and $|t| = 1$, then the integral operators (2.13), (2.14) are unitary ones:*

$$K'_{\mathtt{F}} = K^*{}_{\mathtt{F}} = K^{-1}{}_{\mathtt{F}}. \tag{2.15}$$

Definition 2.8. The operator U_x: $\mathtt{H}_1 \longmapsto \mathtt{G}_1$ is defined by

$$f(x) = U_x e(x) \Leftrightarrow f(x) = \psi(x)e(x), \qquad e(x) \in \mathtt{H}_1, f(x) \in \mathtt{G}_1. \tag{2.16}$$

Likewise, the operator U_y: $\mathtt{H}_2 \longmapsto \mathtt{G}_2$ is determined.

Lemma 2.9. *If the set $\{\psi_n(x)\}_{n=0}^{\infty}$ is an orthonormal basis in the space $\mathtt{H}_1$, then the set $\{\phi_n(x)\}_{n=0}^{\infty}$, defined by (2.8) is an orthonormal basis in the space $\mathtt{G}_1$. Besides, the operator U_x determined by (2.16) is an unitary one*

$$U_x{}^* = U_x{}^{-1}. \tag{2.17}$$

The same affirmation is true for the operator U_y.

Lemma 2.10. *The operators (2.13) and (2.16) satisfy the following relations:*

$$K_{\mathtt{G}} = U_y K_{\mathtt{H}} U_x{}^{-1}, \qquad K_{\mathtt{H}} = U_y{}^{-1} K_{\mathtt{G}} U_x. \tag{2.18}$$

Proof. The proof is trivial. □

2.4. **The Hamiltonian formulation.** From now on we assume that the orthonormal system $\{\varphi_n(x)\}_{n=0}^{\infty}$ is completed in $\mathtt{F}_1 = L^2(R^1;\rho(dx))$. The relations (2.5) indicate a manner by which the position operator $X_{\mathtt{F}_1}$ acts on the elements of this basis in the Fock space $\mathtt{F}_1$. Let us remember ([18]) that the domain $D(X_{\mathtt{F}_1})$ of operator $X_{\mathtt{F}_1}$ is defined by

$$D(X_{\mathtt{F}_1}) = \left\{ f(x) \in \mathtt{F}_1 | \int_{-\infty}^{\infty} |f(x)|^2(1+x^2)\rho(dx) < \infty \right\} \tag{2.19}$$

Using (2.13), (2.14), we define now a momentum operator $P_{\mathtt{F}_1}$, which is conjugate to the position operator $X_{\mathtt{F}_1}$ with respect to the basis $\{\varphi_n(x)\}_{n=0}^{\infty}$ of $\mathtt{F}_1$ in the following way:

$$P_{\mathtt{F}_1} = K_{\mathtt{F}}{}^* Y_{\mathtt{F}_2} K_{\mathtt{F}}. \tag{2.20}$$

Note that a operator $Y_{\mathtt{F}_2}$ in (2.20) is a position operator in the space $\mathtt{F}_2$ defined by analogy with the formulas (2.5). In general , we have ($|t| = 1$)

$$D(P_{\mathtt{F}_1}) = K_{\mathtt{F}}{}^* D(Y_{\mathtt{F}_2}). \tag{2.21}$$

Finally, we define the operator

$$H_{\mathtt{F}_1}(t) = (X_{\mathtt{F}_1})^2 + (P_{\mathtt{F}_1}(t))^2. \tag{2.22}$$

The following theorem is our main result of the present section. The proof is very simple and it is omitted.

Theorem 2.11. *Let a canonical (polynomial) system* $\{\varphi_n(x)\}_{n=0}^{\infty}$ *be completed in the space* $\mathtt{F}_1$*. This system is a set of eigenfunctions of the selfadjoint operator* $H_{\mathtt{F}_1}(t)$ *in* $\mathtt{F}_1$ *defined by (2.22) in* $\mathtt{F}_1$ *if and only if* $t = \pm\imath$*. Moreover, the eigenvalues of the operators* $H_{\mathtt{F}_1}(\pm\imath)$ *are equal to*

$$\lambda_0 = 2b_0^2, \qquad \lambda_n = 2(b_{n-1}^2 + b_n^2), \qquad n \geq 1. \tag{2.23}$$

Remark 2.12. The operator $H_{\mathtt{F}_1} = H_{\mathtt{F}_1}(-\imath)$ is said to be a hamiltonian of the orthonormal system $\{\varphi_n(x)\}_{n=0}^{\infty}$. The domain of the operator $H_{\mathtt{F}_1}$ is obtained from (2.19) and (2.21) by the following formulas

$$D(H_{\mathtt{F}_1}) = \overline{D(X_{\mathtt{F}_1}^2) \cap D(P_{\mathtt{F}_1}^2)}.$$

Also, we denote by

$$P_{\mathtt{F}_1} = P_{\mathtt{F}_1}(-\imath), \quad P_{\mathtt{F}_2} = P_{\mathtt{F}_2}(\imath), \quad H_{\mathtt{F}_2} = H_{\mathtt{F}_2}(\imath). \tag{2.24}$$

The proof of the following lemmas is left to the reader.

Lemma 2.13. *The operators* $X_{\mathtt{F}_1}$, $P_{\mathtt{F}_1}$,$H_{\mathtt{F}_1}$ *act on the basis vectors* $\{\varphi_n(x)\}_{n=0}^{\infty}$ *of the space* $\mathtt{F}_1$ *by*

$$X_{\mathtt{F}_1}\varphi_0(x) = b_0\varphi_1(x), \tag{2.25}$$
$$P_{\mathtt{F}_1}\varphi_0(x) = -\imath b_0\varphi_1(x), \tag{2.26}$$
$$H_{\mathtt{F}_1}\varphi_0(x) = \lambda_0\varphi_0(x), \tag{2.27}$$
$$X_{\mathtt{F}_1}\varphi_n(x) = b_{n-1}\varphi_{n-1}(x) + b_n\varphi_{n+1}(x), \quad n \geq 1, \tag{2.28}$$
$$P_{\mathtt{F}_1}\varphi_n(x) = \imath(b_{n-1}\varphi_{n-1}(x) - b_n\varphi_{n+1}(x)), \quad n \geq 1, \tag{2.29}$$
$$H_{\mathtt{F}_1}\varphi_n(x) = \lambda_n\varphi_n(x), \tag{2.30}$$

where the eigenvalues $\lambda_n, n \geq 0$, *are defined by (2.23).*

Lemma 2.14. *Under the assumptions of the lemma 2.9 the operators (2.22),(2.20) comply with the following relations:*

$$P_{\mathtt{G}_1} = U_x P_{\mathtt{H}_1} {U_x}^{-1}, \qquad H_{\mathtt{G}_1} = U_x H_{\mathtt{H}_1} {U_x}^{-1}. \tag{2.31}$$

Remark 2.15. The previous statement still stands for the operators $(P_{\mathtt{F}_1})(t), (H_{\mathtt{F}_1}))(t)$ at any t ($|t| = 1$).

2.5. **The generalised Fourier transform.** In this subsection we define the Fourier transform conforming to an orthonormal system $\{\varphi_n(x)\}_{n=0}^{\infty}$ in the space $\mathtt{F}_1$(see[19]).

Definition 2.16. Let $\{\varphi_n(x)\}_{n=0}^{\infty}$ be an orthonormal basis in the space $\mathtt{F}_1$. The unitary operators $K_{\mathtt{F}}(\pm\imath)$ are called the generalized (direct and inverse) Fourier transforms. We denote by

$$F_\varphi = K_{\mathtt{F}}(-\imath), \qquad {F_\varphi}^{-1} = K_{\mathtt{F}}(\imath). \tag{2.32}$$

The following theorem can be proved by direct calculations.

Theorem 2.17. *We have in the Hilbert space* $\mathtt{F}_2$ *for the operators (2.20):*

$$P_{\mathtt{F}_2}F_\varphi = F_\varphi X_{\mathtt{F}_1}, \qquad Y_{\mathtt{F}_2}F_\varphi = F_\varphi P_{\mathtt{F}_1}, \qquad P_{\mathtt{F}_1}{F_\varphi}^{-1} = {F_\varphi}^{-1}Y_{\mathtt{F}_2}, \qquad X_{\mathtt{F}_1}{F_\varphi}^{-1} = {F_\varphi}^{-1}P_{\mathtt{F}_2}. \tag{2.33}$$

and for the operators (2.22):

$$H_{\mathtt{F}_2}F_\varphi = F_\varphi H_{\mathtt{F}_1}, \qquad H_{\mathtt{F}_1}{F_\varphi}^{-1} = {F_\varphi}^{-1}H_{\mathtt{F}_2}. \tag{2.34}$$

2.6. **The generalized oscillators algebra.** Let $\{\varphi_n(x)\}_{n=0}^{\infty}$ be an orthonormal basis in the Fock space $\mathtt{F}_1$. We construct some (generalized) oscillators algebra corresponding the system $\{\varphi_n(x)\}_{n=0}^{\infty}$. To this end we define ladder operators $a^+{}_{\mathtt{F}_1}$ and $a^-{}_{\mathtt{F}_1}$ by the usual formulas:

$$a^+_{\mathtt{F}_1} = \frac{1}{\sqrt{2}}\left(X_{\mathtt{F}_1} + \imath P_{\mathtt{F}_1}\right), \qquad a^-_{\mathtt{F}_1} = \frac{1}{\sqrt{2}}\left(X_{\mathtt{F}_1} - \imath P_{\mathtt{F}_1}\right). \tag{2.35}$$

It is readily seen that (for the classical orthogonal polynomials)

$$a^{-\,*}_{\mathtt{F}_1} = a^+_{\mathtt{F}_1}, \qquad a^{+\,*}_{\mathtt{F}_1} = a^-{}_{\mathtt{F}_1}.$$

and

$$D(a^-_{\mathtt{F}_1}) = D(a^+_{\mathtt{F}_1}) = \overline{D(X_{\mathtt{F}_1}) \cap D(P_{\mathtt{F}_1})}.$$

Lemma 2.18. *The action of operators (2.35) on the vectors of the basis in the space* $\mathtt{F}_1$ *is given by the standard formulas:*

$$a^+_{\mathtt{F}_1}\varphi_n(x) = \sqrt{2}b_n\varphi_{n+1}(x), \qquad a^-_{\mathtt{F}_1}\varphi_n(x) = \sqrt{2}b_{n-1}\varphi_{n-1}(x), \qquad n \geq 0. \tag{2.36}$$

It is easy to prove from (2.35), (2.29) and (2.28).

Lemma 2.19. *Under the assumptions of the lemma 2.9 the operators (2.36),(2.16) comply with the following relations:*

$$a^\pm_{\mathtt{G}_1} = U_x a^\pm_{\mathtt{H}_1} U_x{}^{-1}, \qquad [X_{\mathtt{F}_1}, P_{\mathtt{F}_1}] = \imath[a^-_{\mathtt{H}_1}, a^+_{\mathtt{H}_1}]. \tag{2.37}$$

Definition 2.20. An operator $N_{\mathtt{F}_1}$ in the Fock space $\mathtt{F}_1$ equipped with the basis $\{\varphi_n(x)\}_{n=0}^{\infty}$ is called a number operator if it acts on basis vectors by formulas:

$$N_{\mathtt{F}_1}\varphi_n(x) = n\varphi_n(x), \qquad n \geq 0. \tag{2.38}$$

Lemma 2.21. *Under the assumptions of the lemma 2.9 the operators (2.38) satisfy the following relations:*

$$N_{\mathtt{G}_1} = U_x N_{\mathtt{H}_1} U_x{}^{-1}. \tag{2.39}$$

The proof is simple.

Remark 2.22. 1. We denote by $B(N)$ a function of operator N in the space $\mathtt{F}_1$ which acts on the vectors of the basis $\{\varphi_n(x)\}_{n=0}^{\infty}$ by

$$B(N_{\mathtt{F}_1})\varphi_n(x) = b^2_{n-1}\varphi_n(x), \qquad n \geq 0, \quad b_{-1} = 0. \tag{2.40}$$

2. Let the assumptions of the lemma 2.9 be held.Then from (2.40) and the lemma 2.21 it follows that

$$B(N_{\mathtt{G}_1}) = U_x B(N_{\mathtt{H}_1}) U_x{}^{-1}. \tag{2.41}$$

The following theorem is our main result of the present subsection. The proof is very simple and it is omitted.

Theorem 2.23. *Under the assumptions of the lemma 2.9 the operators (2.35), (2.40) in the Fock space* $\mathtt{F}_1$ *satisfy the following relations:*

$$[a^-{}_{\mathtt{F}_1}, a^+{}_{\mathtt{F}_1}] = 2(B(N_{\mathtt{F}_1} + I_{\mathtt{F}_1}) - B(N_{\mathtt{F}_1})), \quad [N_{\mathtt{F}_1}, a^\pm{}_{\mathtt{F}_1}] = \pm a^\pm_{\mathtt{F}_1}. \tag{2.42}$$

Let the sequence $\{b_n\}_{n=0}^{\infty}$ *be defined by (2.2) in the space* $\mathtt{F}_1$ *with the measure* ρ*. If there is a real number* A *and a real function* $C(n)$*, such that this sequence satisfies the following recurrence relation:*

$$b^2_n - Ab^2_{n-1} = C(n), \qquad n \geq 0, \quad b_{-1} = 0, \tag{2.43}$$

then the operators (2.35),(2.40) satisfy the following conditions:

$$a^-_{\mathtt{F}_1} a^+_{\mathtt{F}_1} - A a^+_{\mathtt{F}_1} a^-_{\mathtt{F}_1} = 2C(N_{\mathtt{F}_1}), \tag{2.44}$$

apart from(2.42). Here the function $C(N)$ is defined similarly (2.40) with $C(n)$ instead of ${b_{n-1}}^2$.

Proof. It is follows from the obvious relations:

$$\begin{aligned} a^-_{\mathtt{F}_1} a^+_{\mathtt{F}_1} \varphi_n(x) &= 2b_n^2 \varphi_n(x), \\ a^+_{\mathtt{F}_1} a^-_{\mathtt{F}_1} \varphi_n(x) &= 2b_{n-1}^2 \varphi_n(x), \\ n \geq 0, \quad b_{-1} &= 0. \end{aligned} \tag{2.45}$$

□

Definition 2.24. An algebra A_φ is called a generalized oscillator algebra corresponding to the orthonormal system $\{\varphi_n(x)\}_{n=0}^\infty$ if A_φ is generated by generators $a^\pm_{\mathtt{F}_1}$, $N_{\mathtt{F}_1}$, which satisfy the relations of (2.45) and the two latter ones of (2.42) .

2.7. **The generalized algebra $su_\varphi(2)$.** Let $\mathtt{F_i}, \mathtt{i} = 0, 1$ be the Fock spaces equipped respectively with bases $\{\varphi_n(x_i)\}_{n=0}^\infty$ and $a^\pm_{\mathtt{F}_i}, N_{\mathtt{F}_i}, i = 0, 1$ be the generators of the generalized oscillators algebra A_φ. These generators $a^\pm_{\mathtt{F}_i}, N_{\mathtt{F}_i}, i = 0, 1$ are generators of an algebra of the system of the two independent oscillators if they satisfy the following commutation relations:

$$\begin{aligned} &a^-_{\mathtt{F}_i} a^+_{\mathtt{F}_i} = 2B(N_{\mathtt{F}_i} + I_{\mathtt{F}_i}), \qquad && a^+_{\mathtt{F}_i} a^-_{\mathtt{F}_i} = 2B(N_{\mathtt{F}_i}), \qquad && [N_{\mathtt{F}_i}, a^\pm_{\mathtt{F}_i}] = \pm a^\pm_{\mathtt{F}_i}, \\ &[a^\pm_{\mathtt{F}_1}, a^\pm_{\mathtt{F}_2}] = 0, && [N_{\mathtt{F}_1}, a^\pm_{\mathtt{F}_2}] = 0, && [N_{\mathtt{F}_2}, a^\pm_{\mathtt{F}_1}] = 0. \end{aligned} \tag{2.46}$$

We denote by $su_\varphi(2)$ an algebra generated by the generators $J^\varphi_+, J^\varphi_-, J^\varphi_z$, which are connected with the generators $a^\pm_{\mathtt{F}_i}, N_{\mathtt{F}_i}$ according to the rules:

$$J^\varphi_+ = a^+_{\mathtt{F}_1} a^-_{\mathtt{F}_2}, \quad J^\varphi_- = a^+_{\mathtt{F}_2} a^-_{\mathtt{F}_1}, \quad J^\varphi_z = 2^{-1}(N_{\mathtt{F}_1} - N_{\mathtt{F}_2}). \tag{2.47}$$

Theorem 2.25. *Let the function $B(x)$ defined by (2.40) is a solution to the following equation:*

$$f(x)f(y+1) - f(y)f(x+1) = f(x-y). \tag{2.48}$$

Then the operators $J^\varphi_+, J^\varphi_-, J^\varphi_z$ in the space $\mathtt{F}_1 \otimes \mathtt{F}_2$ obey to the following commutation relations:

$$[J^\varphi_z, J^\varphi_\pm] = \pm J^\varphi_\pm, \qquad [J^\varphi_+, J^\varphi_-] = 2B(J^\varphi_z). \tag{2.49}$$

The proof is by direct calculation.

Remark 2.26. We see at once that the relations (2.49) are the extensions of the usual commutation relations of the algebra $su(2)$ and reduce to the latter in the case $B(x) = x$. An algebra generated by the generators $J^\varphi_\pm, J^\varphi_z$ complying with (2.49) is called a deformed algebra $SU_q(2)$ corresponding to the orthonormal system $\{\varphi_n(x)\}_{n=0}^\infty$. Indeed, it follows from the next lemma that all solutions to the equation (2.48) make up an one-parameter family with the parameter q.

Lemma 2.27. *If a function $f(x)$ is analytical in the region $|x| < R$, where $R > 1$, and satisfies to the equation (2.48), then one can represent it in the following form:*

$$f(x) = \frac{sinh(\eta x)}{sinh(\eta)}, \qquad exp(\eta) = q. \tag{2.50}$$

The proof is left to the reader.

Remark 2.28. 1. The following functions:

$$B(x) = x, \qquad B(x) = [x, q] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2.51}$$

give us some examples of solutions to the equation (2.48). Note that the solution $B(x) = x$ corresponds to the usual harmonic oscillator and to the algebra $su(2)$; in the case $B(x) = [x, q]$ we obtain the deformed oscillator ([8],[9]) and the quantum group $SU_q(2)$.

2. If $B(x)$ does not a solution to (2.48), then the second of commutation relation (2.49) takes the form

$$[J_+^\varphi, J_-^\varphi] = (J_z^\varphi)F(J_z^\varphi, C_z), \tag{2.52}$$

Here $C_z = N_1 + N_2$ is a element of the center of the algebra $su_\varphi(2)$ generated by $J_\pm^\varphi, J_z^\varphi$ complying with (2.52) and the first of relation (2.49). The function F in the right-hand side (2.52) is an analytical function in its own arguments.

2.8. In this subsection we will provide the following answer. How is a measure μ to be so that the momentum operator $P_{\mathtt{G}_1}$ satisfies

$$P_{\mathtt{G}_1}(uv) = vP_{\mathtt{G}_1}(u) + uP_{\mathtt{G}_1}(v). \tag{2.53}$$

Let $\mathtt{G}_1 \subset \mathtt{H}_1$ and the set $\mathtt{G}_1$ be dense in the space $\mathtt{H}_1$. Denote by $\overline{P}_{\mathtt{G}_1}$ the closure of the momentum operator $P_{\mathtt{G}_1}$ in $\mathtt{H}_1$. It is easy to prove the next theorem.

Theorem 2.29. *The operator $\overline{P}_{\mathtt{G}_1}$ in $\mathtt{H}_1$ satisfy (2.53) if and only if the following conditions are held:*

$$\begin{aligned} &\overline{P}_{\mathtt{G}_1} = \imath\sqrt{2}a^-{}_{\mathtt{H}_1}, \\ &{b_n}^2 = (n+1){b_0}^2, \quad n \geq 1. \end{aligned} \tag{2.54}$$

Remark 2.30. The second of condition (2.54) means that an appropriate oscillator is the usual quantum mechanical one.

Below we consider the examples of generalized oscillators algebras corresponding to the classical orthogonal polynomials.

3. Hermite polynomials

3.1. First we consider the main example underlying our construction, namely, the Hermite polynomials ([14],[15],[16]).

Let $\mathtt{G}_1 = L^2(R)$, $\mathtt{H}_1 = L^2(R; \frac{1}{\sqrt{\pi}}\exp(-x^2)dx$ and

$$\psi(x) = \pi^{-\frac{1}{4}} \exp(-\frac{x^2}{2}). \tag{3.1}$$

We denote by $H_n(x)$ the Hermite polynomials

$$H_n(x) = n! \sum_{\nu=0}^{[\frac{n}{2}]} \frac{(-1)^\nu}{\nu!} \frac{(2x)^{n-2\nu}}{(n-2\nu)!}, \tag{3.2}$$

We define the functions $\{\psi_n(x)\}_{n=0}^\infty$ and $\{\phi_n(x)\}_{n=0}^\infty$ by the following formulas:

$$\psi_n(x) = \sqrt[4]{\pi}{d_n}^{-1}H_n(x), \qquad \phi_n(x) = {d_n}^{-1}\exp(-\frac{x^2}{2})H_n(x), \qquad n \geq 0, \tag{3.3}$$

where

$$d_n = (2^n n!\sqrt{\pi})^{\frac{1}{2}}, \qquad n \geq 0. \tag{3.4}$$

The recurrence relations for the Hermite polynomials ([14]) give us the formulas (2.5) ,(2.6) with

$$b_n = \frac{1}{2}(\frac{d_{n+1}}{d_n}) = \sqrt{\frac{n+1}{2}}. \tag{3.5}$$

From the Mehler formula for the Hermite polynomials ([10]) the following expression for the Poisson kernel follows:

$$\pi^{-\frac{1}{2}}\sum_{n=0}^{\infty}\omega^n \cdot \psi_n(x)\cdot\psi_n(y) = (1-\omega^2)^{-\frac{1}{2}}\exp(\frac{2xy\omega-(x^2+y^2)\omega^2}{1-\omega^2}). \tag{3.6}$$

Combining (3.6) with the definition of the (direct and inverse) generalized Fourier transform conforming to the orthonormal system $\{\varphi_n(x)\}_{n=0}^{\infty}$ we get

$$F_\phi = K_{\mathtt{G}}(-\imath), \qquad {F_\phi}^{-1} = K_{\mathtt{G}}(\imath), \tag{3.7}$$

where respectively

$$\mathfrak{K}_{\mathtt{F}}(x,y;-\imath) = \frac{\exp(-\imath xy)}{\sqrt{2\pi}}, \qquad \mathfrak{K}_{\mathtt{F}}(x,y;\imath)) = \frac{\exp(\imath xy)}{\sqrt{2\pi}}. \tag{3.8}$$

Let us remark that in this case the generalized Fourier transform be the same as the usual Fourier transform. An easy computation shows that we have in the space $\mathtt{H}_1$:

$$\overline{P_{\mathtt{G}_1}} = \imath\frac{d}{dx}, \tag{3.9}$$

that is conforming to the theorem 2.29 since the conditions (2.54) are valid. Note also that

$$\begin{aligned} {a^+}_{\mathtt{G}_1} &= \frac{1}{\sqrt{2}}\left(X_{\mathtt{G}_1} - \frac{d}{dx}\right), \qquad {a^-}_{\mathtt{G}_1} = \frac{1}{\sqrt{2}}\left(X_{\mathtt{G}_1} + \frac{d}{dx}\right), \\ H_{\mathtt{G}_1} &= (X_{\mathtt{G}_1})^2 + (P_{\mathtt{G}_1})^2 = -\frac{d^2}{dx^2} + x^2. \end{aligned} \tag{3.10}$$

Then the equation

$$H_{\mathtt{G}_1}\phi_n(x) = \lambda_n\phi_n(x), \quad \lambda_n = 2n+1, \tag{3.11}$$

takes the following form:

$$(-\frac{d^2}{dx^2} + x^2)\phi_n(x) = (2n+1)\phi_n(x). \tag{3.12}$$

It can easily be checked that (3.12) is equivalent to the well-known equation for the Hermite polynomials:

$${H_n}''(x) - 2x{H_n}'(x) + 2nH_n(x) = 0. \tag{3.13}$$

In the next section we present the first substantive example, namely, the ultraspherical polynomials.

4. Ultrasferical polynomials

First we consider a particular case of the ultraspherical polynomials, namely, the Legendre polynomials.

4.1. **The Legendre polynomials.** Let

$$\mathtt{G}_1 = L^2([-1,1]), \quad \mathtt{H}_1 = L^2([-1,1];2^{-1}),$$

and the function $\psi(x) = \frac{1}{\sqrt{2}}$. The Legendre polynomials are defined by

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right). \tag{4.1}$$

The functions of the orthonormal systems $\{\psi_n(x)\}_{n=0}^{\infty}$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ are given by the following formulas:

$$\psi_n(x) = \sqrt{2}\phi_n(x), \qquad \phi_n(x) = \sqrt{\frac{2n+1}{2}}P_n(x), \qquad n \geq 0. \tag{4.2}$$

Taking into account the recurrence relations for the Legendre polynomials ([14]) we obtain the formulas (2.5) ,(2.6) where

$$b_n = \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}, \quad n \geq 0, \tag{4.3}$$

and

$$b_n^2 - b_{n-1}^2 = -(2n-1)^{-1}(2n+1)^{-1}(2n+3)^{-1}. \tag{4.4}$$

In the construction of the momentum operator we will use the following differential operator:

$$A = (1-x^2)\frac{d}{dx}, \tag{4.5}$$

in the space $\mathtt{G}_1$. The operator A acts on the basis vectors $\{\phi_n(x)\}_{n=0}^{\infty}$ of $\mathtt{G}_1$ by:

$$A\phi_n(x) = (n+1)b_{n-1}\phi_{n-1}(x) - nb_n\phi_{n+1}(x), \qquad n \geq 0, \quad b_{-1} = 0. \tag{4.6}$$

Using the definition (2.38) of the number operator N and (2.35), from (2.5) ,(2.6) and (4.5) ,(4.6) we have

$$\begin{gathered} P_{\mathtt{H}_1} = \imath(N_{\mathtt{H}_1} - (2^{-1})I_{\mathtt{H}_1})^{-1}(N_{\mathtt{H}_1} + 3(2^{-1})I_{\mathtt{H}_1})^{-1} \\ ((N_{\mathtt{H}_1} + (2^{-1})I_{\mathtt{H}_1})A - X_{\mathtt{H}_1}N_{\mathtt{H}_1} - (2^{-1})((N_{\mathtt{H}_1} + 3(2^{-1})I_{\mathtt{H}_1})X_{\mathtt{H}_1}). \end{gathered} \tag{4.7}$$

It can easily be checked that the formula (2.29) for the operator $P_{\mathtt{H}_1}$ is valid:

$$P_{\mathtt{H}_1}\phi_n(x) = \imath(b_{n-1}\phi_{n-1}(x) - b_n\phi_{n+1}(x)), \qquad n \geq 0, \quad b_{-1} = 0. \tag{4.8}$$

Then the ladder operators $a^-_{\mathtt{H}_1}$ and $a^+_{\mathtt{H}_1}$ are given by:

$$\begin{gathered} a^-_{\mathtt{H}_1} = \frac{1}{\sqrt{2}}(N_{\mathtt{H}_1} + 3(2^{-1})I_{\mathtt{H}_1})^{-1}(A + X_{\mathtt{H}_1}N_{\mathtt{H}_1}), \\ a^+_{\mathtt{H}_1} = \frac{1}{\sqrt{2}}(N_{\mathtt{H}_1} - (2^{-1})I_{\mathtt{H}_1})^{-1}(-A + X_{\mathtt{H}_1}(N_{\mathtt{H}_1} + I_{\mathtt{H}_1})),. \end{gathered} \tag{4.9}$$

Further, the eigenvalue of the operator $H_{\mathtt{H}_1} = (X_{\mathtt{H}_1})^2 + (P_{\mathtt{H}_1})^2$ (the energy levels) amount to:

$$\lambda_0 = \frac{2}{3}, \quad \lambda_n = \frac{n(n+1) - 2^{-1}}{(n+3(2^{-1}))(n-(2^{-1}))}, \quad n > 0. \tag{4.10}$$

Theorem 4.1. *The equation $H_{\mathtt{H}_1}\phi_n(x) = \lambda_n\phi_n(x)$, $\quad n \geq 0$, where the eigenvalue λ_n of the operator $H_{\mathtt{H}_1} = (X_{\mathtt{H}_1})^2 + (P_{\mathtt{H}_1})^2$ defined by (4.10), is equivalent to the usual differential equation for the Legendre polynomials ([12]) :*

$$\frac{d}{dx}((1-x^2)\frac{d}{dx})P_n(x) + n(n+1)P_n(x) = 0, \quad n \geq 0. \tag{4.11}$$

Proof. On account of (4.10) we rewrite the equation $H_{\mathtt{H}_1}\phi_n(x)=\lambda_n\phi_n(x), \quad n\geq 0$, as an operator equality in the space $\mathtt{H}_1$:

$$\begin{aligned}(X_{\mathtt{H}_1})^2+(P_{\mathtt{H}_1})^2=(N_{\mathtt{H}_1}(N_{\mathtt{H}_1}+I_{\mathtt{H}_1})-(2^{-1})I_{\mathtt{H}_1})\\ (N_{\mathtt{H}_1}-(2^{-1})I_{\mathtt{H}_1})^{-1}(N_{\mathtt{H}_1}+3(2^{-1})I_{\mathtt{H}_1})^{-1}.\end{aligned}\tag{4.12}$$

Substituting (4.7) in (4.12) we get

$$\begin{gathered}(2^{-1})I_{\mathtt{H}_1}+\imath((N_{\mathtt{H}_1}+(2^{-1})I_{\mathtt{H}_1})A-X_{\mathtt{H}_1}N_{\mathtt{H}_1}-\\ (2^{-1})((N_{\mathtt{H}_1}+3(2^{-1})I_{\mathtt{H}_1})X_{\mathtt{H}_1})P_{\mathtt{H}_1}+(N_{\mathtt{H}_1}-(2^{-1})I_{\mathtt{H}_1})\\ (N_{\mathtt{H}_1}+3(2^{-1})I_{\mathtt{H}_1})X_{\mathtt{H}_1}^2=N_{\mathtt{H}_1}(N_{\mathtt{H}_1}+I_{\mathtt{H}_1}).\end{gathered}\tag{4.13}$$

It is not hard to prove that

$$\begin{gathered}-A^2+{X_{\mathtt{H}_1}}^2N_{\mathtt{H}_1}(N_{\mathtt{H}_1}+I_{\mathtt{H}_1})=(2^{-1})I_{\mathtt{H}_1}+\imath((N_{\mathtt{H}_1}+(2^{-1})I_{\mathtt{H}_1})A\\ -X_{\mathtt{H}_1}N_{\mathtt{H}_1}-(2^{-1})((N_{\mathtt{H}_1}+3(2^{-1})I_{\mathtt{H}_1})X_{\mathtt{H}_1})P_{\mathtt{H}_1}+\\ (N_{\mathtt{H}_1}-(2^{-1})I_{\mathtt{H}_1})(N_{\mathtt{H}_1}+3(2^{-1})I_{\mathtt{H}_1})X_{\mathtt{H}_1}^2.\end{gathered}\tag{4.14}$$

Then from (4.7) and (4.14) we have

$$(A^2+(I_{\mathtt{H}_1}-X_{\mathtt{H}_1}^2)n(n+1))\phi_n(x)=0.\tag{4.15}$$

Substituting (4.5) and (4.2) in (4.15) we get the equation (4.11). □

Remark 4.2. Using (4.15) one can get

$$(N_{\mathtt{H}_1}+(2^{-1})I_{\mathtt{H}_1})=D=(2^{-1})(I_{\mathtt{H}_1}-4\frac{d}{dx}((1-x^2)\frac{d}{dx}))^{\frac{1}{2}}.\tag{4.16}$$

We exclude the number operator $N_{\mathtt{H}_1}$ from the right-side of (4.7). Then we obtain

$$P_{\mathtt{H}_1}=\imath(D^2-I_{\mathtt{H}_1})^{-1}(DA-X_{\mathtt{H}_1}D-(2^{-1})DX_{\mathtt{H}_1}),\tag{4.17}$$

$${a^-}_{\mathtt{H}_1}=\frac{1}{\sqrt{2}}(D+I_{\mathtt{H}_1})^{-1}(A+X_{\mathtt{H}_1}(D-(2^{-1})I_{\mathtt{H}_1})),\tag{4.18}$$

$${a^+}_{\mathtt{H}_1}=\frac{1}{\sqrt{2}}(D-I_{\mathtt{H}_1})^{-1}(-A+X_{\mathtt{H}_1}(D+(2^{-1})I_{\mathtt{H}_1})).\tag{4.19}$$

Furthermore

$$\begin{gathered}H_{\mathtt{H}_1}=(D^2-\frac{3}{4}I_{\mathtt{H}_1})(D^2-I_{\mathtt{H}_1})^{-1}=\\ I_{\mathtt{H}_1}+\frac{1}{4}(D^2-I_{\mathtt{H}_1})^{-1}=I_{\mathtt{H}_1}-(3I_{\mathtt{H}_1}+\frac{d}{dx}((1-x^2)\frac{d}{dx}))^{-1}.\end{gathered}\tag{4.20}$$

and in view of (4.4) the commutation relations (2.42) for the operators (4.18)-(4.19) will look like:

$$[{a^-}_{\mathtt{H}_1},{a^+}_{\mathtt{H}_1}]=-\frac{1}{4}({N_{\mathtt{H}_1}}^2-\frac{1}{4}I_{\mathtt{H}_1})^{-1}({N_{\mathtt{H}_1}}^2+\frac{3}{2}I_{\mathtt{H}_1})^{-1}.\tag{4.21}$$

Now we turn to the general case of the ultraspherical polynomials.

4.2. **The Gegenbauer polynomials.** Let

$$\mathtt{G}_1 = L^2([-1,1]),$$

$$\mathtt{H}_1 = L^2([-1,1];(d_0(\alpha))^{-2}(1-x^2)^{\alpha}dx),$$

where

$$d_0{}^2(\alpha) = 2^{2\alpha+1}\frac{(\Gamma(\alpha+1))^2}{\Gamma(2(\alpha+1))}.$$

The ultraspherical polynomials are defined by the hypergeometric function ([21, 22]):

$$P_n^{(\alpha,\alpha)}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1\left(-n, n+2\alpha+1; \alpha+1; \frac{1-x}{2}\right). \tag{4.22}$$

The Pochhammer-symbol ([20]) $(\beta)_n$ is defined by $(\beta)_0 = 1, \quad (\beta)_n = \beta(\beta+1)\cdots(\beta+n-1), \quad n \geq 1$. For $\alpha > -1$ the following orthogonal relations are valid:

$$\int_{-1}^{1} P_n^{(\alpha,\alpha)}(x)P_m^{(\alpha,\alpha)}(x)(1-x^2)^{\alpha}dx = d_n{}^2\delta_{mn}, \quad n,m \geq 0,$$

with the constant of normalization d_n given by

$$d_n{}^2 = \frac{2^{2\alpha+1}(\Gamma(n+\alpha+1))^2}{(2n+2\alpha+1)n!\Gamma(n+2\alpha+1)}, \quad n \geq 0. \tag{4.23}$$

The Gegenbauer polynomials are defined as usual ([14]):

$$P_n^{(\lambda)}(x) = \frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)}P_n^{(\alpha,\alpha)}(x), \qquad \alpha = \lambda - 2^{-1}, \tag{4.24}$$

$(\lambda > -2^{-1}, \quad n \geq 0)$.

Let $\psi(x) = d_0{}^{-1}(1-x^2)^{2^{-1}\alpha}$. We determine a functions of the orthonormal systems $\{\psi_n(x)\}_{n=0}^{\infty}$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ by the following formulas:

$$\psi_n(x) = d_0 d_n{}^{-1}P_n^{(\alpha,\alpha)}(x), \qquad \phi_n(x) = \psi(x)\psi_n(x), \qquad n \geq 0, \tag{4.25}$$

where d_n is given via (4.23). The function $\psi_n(x)$ defined by (4.22) satisfies the relations (2.5) and (2.6), where

$$b_n = \sqrt{\frac{(n+1)(n+2\alpha+1)}{(2n+2\alpha+1)(2n+2\alpha+3)}}, \quad n \geq 0, \quad b_{-1} = 0, \tag{4.26}$$

and

$$b_n{}^2 - b_{n-1}{}^2 = 2^{-1}\lambda(\lambda-1)(n+\lambda)^{-1}(n-1+\lambda)^{-1}(n+\lambda+1)^{-1}. \tag{4.27}$$

In order to find a differential expression for the momentum operator $P_{\mathtt{H}_1}$ we use the known formula ([14]):

$$A\psi_n(x) = (n+2\alpha+1)b_{n-1}\psi_{n-1}(x) - nb_n\psi_{n+1}(x), \qquad n \geq 0, \quad b_{-1} = 0, \tag{4.28}$$

where A and b_n are defined by (4.5) and (4.25) respectively. Combining (4.26) and (2.5) with (4.25) we get

$$\begin{aligned} a_{\mathtt{H}_1}^{-} &= \frac{1}{\sqrt{2}}(N_{\mathtt{H}_1} + (\alpha + 3(2^{-1}))I_{\mathtt{H}_1})^{-1}(A + X_{\mathtt{H}_1}N_{\mathtt{H}_1}), \\ a_{\mathtt{H}_1}^{+} &= \frac{1}{\sqrt{2}}(N_{\mathtt{H}_1} + (\alpha - (2^{-1}))I_{\mathtt{H}_1})^{-1}(-A + X_{\mathtt{H}_1}(N_{\mathtt{H}_1} + (2\alpha+1)I_{\mathtt{H}_1})), \end{aligned} \tag{4.29}$$

which generalize (4.9) and reduce to these as $\alpha = 0$. From (4.29) and the formula $P_{\mathtt{H}_1} = \frac{1}{\imath\sqrt{2}}(a^{+}{}_{\mathtt{H}_1} - a^{-}{}_{\mathtt{H}_1})$ it follows that:

$$\begin{aligned}P_{\mathtt{H}_1} = \imath(N_{\mathtt{H}_1} + (\alpha - (2^{-1}))I_{\mathtt{H}_1})^{-1}((N_{\mathtt{H}_1} + (\alpha + 3(2^{-1}))I_{\mathtt{H}_1})^{-1}\\ (N_{\mathtt{H}_1} + (\alpha + (2^{-1}))I_{\mathtt{H}_1})A - (N_{\mathtt{H}_1} + (\alpha + 3(2^{-1}))I_{\mathtt{H}_1})^{-1}\\ X_{\mathtt{H}_1}N_{\mathtt{H}_1} - (\alpha + (2^{-1}))X_{\mathtt{H}_1}).\end{aligned} \tag{4.30}$$

Note that (4.30) generalize (4.7) for the Legendre polynomials and reduce to these as $\alpha = 0$.

Remark 4.3. The energy operator $H_{\mathtt{H}_1} = (X_{\mathtt{H}_1})^2 + (P_{\mathtt{H}_1})^2$ is bounded and has the energy levels

$$\lambda_0 = \frac{2}{2\alpha + 3}, \quad \lambda_n = \frac{n(n + 2\alpha + 1) + (\alpha - 2^{-1})}{(n + \alpha + 3(2^{-1}))(n + \alpha - (2^{-1}))}, \quad n > 0. \tag{4.31}$$

The next theorem is the extension of the analogous theorem 4.1.

Theorem 4.4. *The equation $H_{\mathtt{H}_1}\phi_n(x) = \lambda_n\phi_n(x)$, $n \geq 0$, where the eigenvalues λ_n of the operator $H_{\mathtt{H}_1} = (X_{\mathtt{H}_1})^2 + (P_{\mathtt{H}_1})^2$ are defined by (4.31), is equivalent to the usual differential equation for the ultraspherical polynomials ([12]) :*

$$\begin{aligned}(1 - x^2)\frac{d}{dx}((1 - x^2)^{\alpha - 1}\frac{d}{dx})P_n^{(\alpha,\alpha)}(x) +\\ n(n + 2\alpha + 1)(1 - x^2)^{\alpha + 1}P_n^{(\alpha,\alpha)}(x) = 0,\end{aligned} \tag{4.32}$$

$(n \geq 0)$.

A slight change in the proof of the theorem 4.1 shows that the theorem 4.4 is true.

Remark 4.5. Similarly to (4.17)-(4.19) we have the following formulas:

$$P_{\mathtt{H}_1} = \imath({D_{\mathtt{H}_1}}^2 - I_{\mathtt{H}_1})^{-1}(D_{\mathtt{H}_1}A - X_{\mathtt{H}_1}D_{\mathtt{H}_1} - (\alpha + 2^{-1})D_{\mathtt{H}_1}X_{\mathtt{H}_1}), \tag{4.33}$$

$$a^{-}_{\mathtt{H}_1} = \frac{1}{\sqrt{2}}(D_{\mathtt{H}_1} + I_{\mathtt{H}_1})^{-1}(A + X_{\mathtt{H}_1}(D_{\mathtt{H}_1} - (\alpha + 2^{-1})I_{\mathtt{H}_1})), \tag{4.34}$$

$$a^{+}_{\mathtt{H}_1} = \frac{1}{\sqrt{2}}(D_{\mathtt{H}_1} - I_{\mathtt{H}_1})^{-1}(-A + X_{\mathtt{H}_1}(D_{\mathtt{H}_1} + (\alpha + 2^{-1})I_{\mathtt{H}_1})). \tag{4.35}$$

Analogy with (4.20) gives us:

$$\begin{aligned}H_{\mathtt{H}_1} &= ({D_{\mathtt{H}_1}}^2 - (\alpha^2 + \frac{3}{4})I_{\mathtt{H}_1})({D_{\mathtt{H}_1}}^2 - I_{\mathtt{H}_1})^{-1}\\ &= I_{\mathtt{H}_1} + (\frac{1}{4} - \alpha^2)({D_{\mathtt{H}_1}}^2 - I_{\mathtt{H}_1})^{-1}\\ &= I_{\mathtt{H}_1} + (\frac{1}{4} - \alpha^2)((-\frac{3}{4} + \alpha^2(1 - x^2)^{-1})I_{\mathtt{H}_1} - \frac{d}{dx}((1 - x^2)\frac{d}{dx}))^{-1}.\end{aligned} \tag{4.36}$$

In view of (4.27) the commutation relations (2.42) for the operators (4.29) will look like:

$$[a^{-}_{\mathtt{H}_1}, a^{+}_{\mathtt{H}_1}] = \lambda(\lambda - 1)((N_{\mathtt{H}_1} + \alpha I_{\mathtt{H}_1})^2 - \frac{1}{4}I_{\mathtt{H}_1})^{-1}(N_{\mathtt{H}_1} + (\alpha + \frac{3}{2})I_{\mathtt{H}_1})^{-1}. \tag{4.37}$$

In conclusion of this section we consider yet another special case, namely, the Chebyshev polynomials.

4.3. **Chebyshev polynomials.** The Chebyshev polynomials of the first kind $T_n(x)$ and those of the second kind $U_n(x)$ are special cases of the Gegenbauer polynomials for $\lambda = 0 (\alpha = -2^{-1})$ and $\lambda = 1$ $(\alpha = 2^{-1})$ respectively. In both cases, it follows from (4.27):

$$b_{-1} = 0, \quad b_n = \frac{1}{2}, \quad n \geq 0.$$

Hence all operators - co-ordinate, momentum and hamiltonian - are the same for the Chebyshev polynomials of the first kind, as for those of the second kind. Thus both of these polynomials systems give us the unitary equivalent representations of the same oscillator in the different spaces:

$$\mathtt{H}_1 = L^2([-1,1]; (\sqrt{1-x^2})^{-1} \frac{dx}{{d_0}^2(2^{-1})}),$$

and

$$\mathtt{H}_2 = L^2([-1,1]; \sqrt{1-x^2} \frac{dx}{{d_0}^2(-2^{-1})}).$$

In both cases the energy levels are equal:

$$\lambda_0 = \frac{1}{2}, \quad \lambda_n = 1, \quad n \geq 1.$$

Now we consider the following orthonormal systems in the spaces $\mathtt{H}_1$ and $\mathtt{H}_2$ respectively:

$$\psi_n^{(1)} = \sqrt{2n} \frac{\Gamma(n+\frac{1}{2})^2}{n!\Gamma(n)} 2^{-2n} C_{2n}^n T_n(x), \tag{4.38}$$

$$T_n(x) = 2^{2n} C_{2n}^{n\ -1} {P_n}^{(-2^{-1},-2^{-1})}(x), \tag{4.39}$$

$$\psi_n^{(2)} = \sqrt{2n+2} \frac{\Gamma(n+3\frac{1}{2})^2}{n!\Gamma(n+2)} 2^{-2(n+1)} C_{2(n+1)}^{n+1} U_n(x), \tag{4.40}$$

$$U_n(x) = 2^{2n+1} {C_{2(n+1)}^{n+1}}^{-1} {P_n}^{(2^{-1},2^{-1})}(x), \tag{4.41}$$

Then the operator A acts on basis vectors in the mentioned spaces by

$$A\psi_n^{(1)} = nb_{n-1}\psi_{n-1}^{(1)} - nb_n\psi_{n+1}^{(1)}, \qquad n \geq 0, \quad b_{-1} = 0. \tag{4.42}$$

$$A\psi_n^{(2)} = (n+2)b_{n-1}\psi_{n-1}^{(2)} - nb_n\psi_{n+1}^{(2)}, \qquad n \geq 0, \quad b_{-1} = 0. \tag{4.43}$$

As before, we have

$$a_{\mathtt{H}_1}^- = \frac{1}{\sqrt{2}}(N_{\mathtt{H}_1} + I_{\mathtt{H}_1})^{-1}(A + X_{\mathtt{H}_1} N_{\mathtt{H}_1}), \tag{4.44}$$

$$a_{\mathtt{H}_1}^+ = \frac{1}{\sqrt{2}}(N_{\mathtt{H}_1} - I_{\mathtt{H}_1})^{-1}(-A + X_{\mathtt{H}_1} N_{\mathtt{H}_1}), \tag{4.45}$$

$$a_{\mathtt{H}_2}^- = \frac{1}{\sqrt{2}}(N_{\mathtt{H}_2} + I_{\mathtt{H}_2})^{-1}(A + X_{\mathtt{H}_2} N_{\mathtt{H}_2}), \tag{4.46}$$

$$a_{\mathtt{H}_2}^+ = \frac{1}{\sqrt{2}}{N_{\mathtt{H}_2}}^{-1}(-A + X_{\mathtt{H}_2}(N_{\mathtt{H}_2} + 2I_{\mathtt{H}_2}), \tag{4.47}$$

One can present the momentum operators in the following forms:

$$P_{\mathtt{H}_1} = \imath(N_{\mathtt{H}_1} - I_{\mathtt{H}_1})^{-1}((N_{\mathtt{H}_1} + I_{\mathtt{H}_1})^{-1} N_{\mathtt{H}_1} A - (N_{\mathtt{H}_1} + I_{\mathtt{H}_1})^{-1} X_{\mathtt{H}_1} N_{\mathtt{H}_1}). \tag{4.48}$$

$$P_{\mathtt{H}_2} = \imath(N_{\mathtt{H}_2})^{-1}((N_{\mathtt{H}_2} + 2I_{\mathtt{H}_2})^{-1}(N_{\mathtt{H}_2} + I_{\mathtt{H}_2})A - (N_{\mathtt{H}_2} + 2I_{\mathtt{H}_2})^{-1} X_{\mathtt{H}_2} N_{\mathtt{H}_2} - X_{\mathtt{H}_2}). \tag{4.49}$$

5. Nonsymmetric Jacobi matrix

Let μ be a symmetric probability measure on R^1. We construct in this section a non-canonical orthogonal polynomial system $\{\varphi_n(x)\}_{n=0}^{\infty}$ in the space $\mathtt{H}_1$. As before, we define μ_{2k} by (2.3) and look for the positive sequences $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ as solutions of the following equation system:

$$\mu_0 = 1, \quad \mu_{2k} = b_0 c_1 \cdot (b_0 c_1 + b_1 c_2) \cdots \sum_{j=0}^{k-1} b_j c_{j+1}, \qquad k = 0, 1, \dots . \tag{5.1}$$

Contrary to (2.2) there is an infinite number of solutions of the system (5.1). We can find uniquely from (5.1) only

$$d_j = \sqrt{b_j c_{j+1}}, \qquad j = 0, 1, \dots . \tag{5.2}$$

Now we determine a polynomials system $\{\varphi_n(x)\}_{n=0}^{\infty}$ from the given sequences $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ by the following recurrence relations:

$$x\varphi_n(x) = b_n \varphi_{n+1}(x) + c_n \varphi_{n-1}(x), \qquad n \geq 0, \quad b_{-1} = 0, \tag{5.3}$$

$$\varphi_0(x) = 1. \tag{5.4}$$

The canonical polynomial system $\{\psi_n(x)\}_{n=0}^{\infty}$ is determined by the recurrence relations (2.5), (2.6) replacing b_n with d_n. From the theorem 2.3 it follows that the set $\{\psi_n(x)\}_{n=0}^{\infty}$ is a orthonormal system in the space $\mathtt{H}_1$. It is clear that making the following renormalization of the system $\{\varphi_n(x)\}_{n=0}^{\infty}$

$$\varphi_n(x) = \gamma_n \psi_n(x), \qquad n \geq 0, \tag{5.5}$$

with

$$\gamma_0 = 1, \quad \gamma_n = \sqrt{\frac{c_1 \cdot c_2 \cdots c_n}{b_0 \cdot b_2 \cdots b_{n-1}}}, \quad n \geq 1, \tag{5.6}$$

we get from (5.3) the relations (2.5), (2.6) replacing b_n with d_n. Taking into account the orthonormal condition for $\{\psi_n(x)\}_{n=0}^{\infty}$ we obtain the following orthogonal condition for $\{\varphi_n(x)\}_{n=0}^{\infty}$:

$$\int_{-\infty}^{\infty} \varphi_n(x)\varphi_m(x)\mu(dx) = \gamma_n \gamma_m \int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)\mu(dx) == {\gamma_n}^2 \delta_{nm}. \tag{5.7}$$

Therefore the set $\{\psi_n(x)\}_{n=0}^{\infty}$ is an orthogonal system, however it does not need to be a orthonormal system (as $\gamma_n^2 \neq 1$).

Remark 5.1. This argument shows that for a symmetric probability measure one can reduce the recurrence relations (5.3) to the symmetrical ones (2.5).

6. Nonsymmetric scheme

6.1. Let μ be a probability but not necessarily a symmetric measure on $[a, b] \subset R$, i.e. the conditions (2.1) are incorrect. Denote by $\mathtt{H} = L^2([a, b]; \mu)$ the Hilbert space of the square-integrable functions with respect to the measure μ on $[a, b] \subset R$. Let

$$\mu_0 = 1, \qquad \mu_k = \int_{-\infty}^{\infty} x^k \mu(dx), \qquad k = 0, 1, \dots , \tag{6.1}$$

then look for the real sequences $\{b_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$ as solutions of the following equation system:

$$A_{k,n} = b_n A_{k-1,n+1} + a_n A_{k-1,n} + b_{n-1} A_{k-1,n-1}, \qquad n \geq 0, \qquad b_{-1} = 0, \tag{6.2}$$

also satisfying the conditions:

$$A_{0,0} = 1, \quad A_{k,0} = \mu_k, \quad A_{0,k} = 0, \qquad k \geq 1. \tag{6.3}$$

Lemma 6.1. *There is a unique solution to the system of equations (6.2), (6.3) with respect to the variables* $(a_n, b_n A_{k,n}), n \geq 0, k \geq 0$.

If sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ are given, then we define the canonical polynomial system by the recurrence relation

$$x\psi_n(x) = b_n\psi_{n+1}(x) + a_n\psi_n(x) + b_{n-1}\psi_{n-1}(x), \qquad n \geq 0, \qquad b_{-1} = 0, \tag{6.4}$$

$$\psi_0(x) = 1. \tag{6.5}$$

As before, the remark 2.2 is true.

Theorem 6.2. *Let* $\{\psi_n(x)\}_{n=0}^{\infty}$ *be a real polynomials system defined by (6.4), (6.5), and let* μ *be a probability measure on* $[a,b] \subset R$. *A system of polynomials* $\{\psi_n(x)\}_{n=0}^{\infty}$ *is orthonormal with respect to the measure* μ *on* $[a,b] \subset R$ *if and only if the coefficients* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *involved in the recurrence relations (6.4) are the solution of the system (6.2), (6.3), where* μ_k *are defined by (6.1).*

Remark 6.3. Just as in subsection 2.2 we introduce the Hilbert space $\mathtt{G}_1$ and the set $\{\phi_n(x)\}_{n=0}^{\infty}$. The corollary 2.5 still stands for this system.

6.2. We determine (just as in subsection 2.3) the Poisson kernel in the Hilbert space $\mathtt{F}_1 \otimes \mathtt{F}_2$ and the operators U_x and U_y. The lemmas 2.9 and 2.10 still stand for these operators. As before, one can define the momentum operator $P_{\mathtt{F}_1}$, which is conjugate to the position operator $X_{\mathtt{F}_1}$ with respect to the basis $\{\varphi_n(x)\}_{n=0}^{\infty}$ in $\mathtt{F}_1$, and the symmetric hamiltonian $H_{\mathtt{F}_1}(t)$, which does not have to be a selfadjoint operator. Moreover, the set $\{\varphi_n(x)\}_{n=0}^{\infty}$ does not have to be a set of eigenfunctions of the operator $H_{\mathtt{F}_1}(t)$ at any value t. However one can remedy the situation by using new position and momentum operators.

As in the lemma 2.13 we have

$$X_{\mathtt{F}_1}\varphi_0(x) = b_0\varphi_1(x) + a_0\varphi_0(x), \tag{6.6}$$
$$P_{\mathtt{F}_1}\varphi_0(x) = -(\imath)b_0\varphi_1(x) + a_0\varphi_0(x), \tag{6.7}$$
$$X_{\mathtt{F}_1}\varphi_n(x) = b_{n-1}\varphi_{n-1}(x) + a_n\varphi_n(x) + b_n\varphi_{n+1}(x), \quad n \geq 1, \tag{6.8}$$
$$P_{\mathtt{F}_1}\varphi_n(x) = \imath(b_{n-1}\varphi_{n-1}(x) - b_n\varphi_{n+1}(x)) + a_n\varphi_n(x), n \geq 1. \tag{6.9}$$

From (6.6) - (6.9) it follows that:

$$(X_{\mathtt{F}_1} - P_{\mathtt{H}_1})\psi_0(x) = b_0\psi_1(x), \tag{6.10}$$
$$(X_{\mathtt{F}_1} - P_{\mathtt{H}_1})\psi_n(x) = (1-\imath)b_{n-1}\psi_{n-1}(x) + (1+\imath)b_n\psi_{n+1}(x), \quad n \geq 1. \tag{6.11}$$

When (6.10), (6.11) is compared with (2.25),(2.26),(2.28),(2.29), it is apparent that one can introduce the new position $\widetilde{X}_{\mathtt{F}_1}$ and momentum $\widetilde{P}_{\mathtt{F}_1}$ operators as follows:

$$\widetilde{X}_{\mathtt{F}_1} = Re(X_{\mathtt{F}_1} - P_{\mathtt{F}_1}), \tag{6.12}$$
$$\widetilde{P}_{\mathtt{F}_1} = (-\imath)Im(X_{\mathtt{F}_1} - P_{\mathtt{F}_1}). \tag{6.13}$$

If we replace $X_{\mathtt{F}_1} \longmapsto \widetilde{X_{\mathtt{F}_1}}$ and $P_{\mathtt{F}_1} \longmapsto \widetilde{P}_{\mathtt{F}_1}$, then the formulas (2.25),(2.26),(2.28), (2.29) are valid for the operators $\widetilde{X}_{\mathtt{F}_1}$ and $\widetilde{P}_{\mathtt{F}_1}$.

Lemma 6.4. *Let the operators* $\widetilde{X}_{\mathtt{H}_1}$ *and* $\widetilde{P}_{\mathtt{H}_1}$ *be defined by (6.12), (6.13). Then we have the formula (2.20) (with* $t = \imath$*)*

$$\widetilde{P}_{\mathtt{F}_1} = K_{\mathtt{F}}^* \widetilde{Y}_{\mathtt{F}_2} K_{\mathtt{F}}. \tag{6.14}$$

Now we define the energy operator:

$$\widetilde{H}_{\mathtt{F}_1} = \widetilde{X}_{\mathtt{F}_1}^2 + \widetilde{P}_{\mathtt{F}_1}^2. \tag{6.15}$$

The following theorem is similar to theorem 2.11.

Theorem 6.5. *The operator* $\widetilde{H}_{\mathtt{F}_1}$ *defined by (6.15) is a selfadjoint operator in the space* $\mathtt{F}_1$ *with a orthonormal basis* $\{\varphi_n(x)\}_{n=0}^{\infty}$*. Moreover the set* $\{\varphi_n(x)\}_{n=0}^{\infty}$ *is a eigenfunction system of the operator* $\widetilde{H}_{\mathtt{F}_1}$ *and the eigenvalues of this operator are equal to:*

$$\lambda_0 = 2b_0^2, \qquad \lambda_n = 2(b_{n-1}^2 + b_n^2), \qquad n \geq 1. \tag{6.16}$$

We define the ladder operators:

$$\widetilde{a_{\mathtt{H}_1}^{+}} = \frac{1}{\sqrt{2}}\left(\widetilde{X}_{\mathtt{H}_1} + \imath \widetilde{P}_{\mathtt{H}_1}\right), \qquad \widetilde{a_{\mathtt{H}_1}^{-}} = \frac{1}{\sqrt{2}}\left(\widetilde{X}_{\mathtt{H}_1} - \imath \widetilde{P}_{\mathtt{H}_1}\right), \tag{6.17}$$

If we replace $a_{\mathtt{H}_1}^{\pm} \longmapsto \widetilde{a_{\mathtt{H}_1}^{\pm}}$), then the formulas (2.36) are valid. Moreover the theorem 2.23 is also true.

Remark 6.6. It should be stressed that in this case, too, we succeeded in constructing some oscillator system. However, now the position operator does not have to be an operator of the multiplication on an independent variable.

6.3. Now we consider a nonsymmetric Jacobi matrix of a position operator in Fock representation. Let sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$,$\{c_n\}_{n=0}^{\infty}$ and a sequence $\{A_{k,n}\}_{k,n=0}^{\infty}$ be a solution to the following equation system:

$$A_{k,n} = b_n A_{k-1,n+1} + a_n A_{k-1,n} + c_n A_{k-1,n-1}, \qquad n \geq 0, \qquad b_{-1} = 0, \tag{6.18}$$

satisfying the initial conditions (6.3) too. Contrary to (6.2) there is an infinite family of solution to the system (6.18) , (6.3). We can find uniquely from (6.18) , (6.3) only:

$$d_j = \sqrt{b_j c_{j+1}}, \qquad j = 0, 1, \ldots. \tag{6.19}$$

If the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$,$\{c_n\}_{n=0}^{\infty}$ are given, then we define the polynomial system $\left\{\hat{\psi}_n(x)\right\}_{n=0}^{\infty}$ by:

$$x\hat{\psi}_n(x) = b_n \hat{\psi}_{n+1}(x) + a_n \hat{\psi}_n(x) + c_n \hat{\psi}_{n-1}(x), \qquad n \geq 0, \qquad c_0 = 0, \tag{6.20}$$

$$\hat{\psi}_0(x) = 1. \tag{6.21}$$

If the sequences $\{a_n\}_{n=0}^{\infty}$, $\{d_n\}_{n=0}^{\infty}$, are given, then the canonical polynomial system $\{\psi_n(x)\}_{n=0}^{\infty}$ is defined by the recurrence relations (6.4),(6.5) with d_n instead of b_n. It follows from the theorem 6.2 that the set $\{\psi_n(x)\}_{n=0}^{\infty}$ is an orthonormal polynomials system in the space $\mathtt{H}_1$. It can easily be checked that the renormalizaton:

$$\hat{\psi}_n(x) = \gamma_n \psi_n(x), \qquad n \geq 0, \tag{6.22}$$

where

$$\gamma_0 = 1, \quad \gamma_n = \sqrt{\frac{c_1 \cdot c_2 \cdots c_n}{b_0 \cdot b_1 \cdots b_{n-1}}}, \quad n \geq 1, \tag{6.23}$$

reduce (6.20), (6.21) to the symmetric relations (6.4),(6.5). From the orthonormal conditions for the system $\{\psi_n(x)\}_{n=0}^{\infty}$ we obtain the following orthogonal relations:

$$\int_{-\infty}^{\infty} \hat{\psi}_n(x)\hat{\psi}_m(x)\mu(dx) = \gamma_n\gamma_m \int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)\mu(dx) = {\gamma_n}^2\delta_{nm}. \tag{6.24}$$

Note that the remark 5.1 is true in this case too. A main example of the nonsymmetric scheme for the classical orthogonal polynomials is the Laguerre polynomials.

7. Laguerre polynomials

Denote by $\mathtt{G}_1 = L^2({R^1}_+)$, $\mathtt{H}_1 = L^2({R^1}_+; x^\alpha \exp(-x)dx$ and

$$\psi(x) = x^{\frac{\alpha}{2}} \exp(-\frac{x}{2}). \tag{7.1}$$

We determine the Laguerre polynomials ${L^\alpha}_n(x)$ ([14],[16]):

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}{}_1F_1\left(-n; \alpha+1; x\right). \tag{7.2}$$

Let

$${d_n}^2 = \frac{\Gamma(n+\alpha+1)}{n!}. \tag{7.3}$$

We define also the orthonormal systems $\{\psi_n(x)\}_{n=0}^{\infty}$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ by the following formulas:

$$\psi_n(x) = {d_n}^{-1}L_n^\alpha(x), \qquad \phi_n(x) = \psi(x)\psi_n(x), \qquad n \geq 0. \tag{7.4}$$

Using the recurrence relations for the Laguerre polynomials

$$(n+1)L_{n+1}^\alpha(x) = (2n+\alpha+1-x){L^\alpha}_n(x) + (n+\alpha)L_{n-1}^\alpha(x), \quad n \geq 1, \tag{7.5}$$

$$L_0^\alpha(x) = 1, \qquad L_{-1}^\alpha(x) = 0, \tag{7.6}$$

we have (6.20), where

$$\begin{aligned} b_n &= -(n+1)\frac{d_{n+1}}{d_n}, \\ a_n &= 2n+\alpha+1, \\ c_n &= -(n+\alpha)\frac{d_{n-1}}{d_n}. \end{aligned} \tag{7.7}$$

Finally we obtain from (7.7) and (7.4):

$$\begin{aligned} b_n &= -\sqrt{(n+1)(n+\alpha+1)}, \\ a_n &= 2n+\alpha+1, \\ c_n &= b_{n-1}. \end{aligned} \tag{7.8}$$

We consider a differential operator K, which shall play a large role below. The operator K acts on basis vectors by the following formulas ([14]):

$$K = x\frac{d}{dx} = b_{n-1}\psi_{n-1}(x) + n\psi_n(x), \qquad n \geq 0, \quad b_{-1} = 0. \tag{7.9}$$

From (7.9) and (6.10) , (6.11) we get the formula for the operator $P_{\mathtt{H}_1}$:

$$P_{\mathtt{H}_1} = 2N_{\mathtt{H}_1} + (\alpha+1)I_{\mathtt{H}_1} - \imath(2K - X_{\mathtt{H}_1} + (\alpha+1)I_{\mathtt{H}_1}) \tag{7.10}$$

Combining (7.10) , (6.12) and (6.13) we have:

$$\widetilde{X}_{\mathtt{H}_1} = X_{\mathtt{H}_1} - 2N_{\mathtt{H}_1} - (\alpha+1)I_{\mathtt{H}_1}, \tag{7.11}$$

$$\widetilde{P}_{\mathtt{H}_1} = \imath(2K - X_{\mathtt{H}_1} + (\alpha+1)I_{\mathtt{H}_1}). \tag{7.12}$$

Further, from (7.9) and the definition (2.35) and (2.38) of the operators $a^{\pm}_{\mathtt{H}_1}$, $N_{\mathtt{H}_1}$ it follows that:

$$\widetilde{a^{-}_{\mathtt{H}_1}} = \sqrt{2}(K - N_{\mathtt{H}_1}), \tag{7.13}$$

$$\widetilde{a^{+}_{\mathtt{H}_1}} = \sqrt{2}([N_{\mathtt{H}_1}, X_{\mathtt{H}_1}] + (K - N_{\mathtt{H}_1})). \tag{7.14}$$

In view of

$$[K, X_{\mathtt{H}_1}] = X_{\mathtt{H}_1}, \tag{7.15}$$

$$[K - N_{\mathtt{H}_1}, X_{\mathtt{H}_1}] = 2K + (\alpha + 1)I_{\mathtt{H}_1}. \tag{7.16}$$

it is not hard to prove that (6.17) is true for the operators (7.11)-(7.14). Then one can rewrite (7.11),(7.12) in the form:

$$\widetilde{X}_{\mathtt{H}_1} = 2(K - N_{\mathtt{H}_1}) + [N_{\mathtt{H}_1}, X_{\mathtt{H}_1}], \tag{7.17}$$

$$\widetilde{P}_{\mathtt{H}_1} = (-\imath)[N_{\mathtt{H}_1}, X_{\mathtt{H}_1}]. \tag{7.18}$$

Taking into account (7.17),(7.18) and (7.13),(7.14) one can write the hamiltonian $\widetilde{H}_{\mathtt{H}_1}$ defined by (6.15) in the following form:

$$\begin{gathered}\widetilde{H}_{\mathtt{H}_1} = 4(K - N_{\mathtt{H}_1})^2 + 2([N_{\mathtt{H}_1}, X_{\mathtt{H}_1}](K - N_{\mathtt{H}_1}) + \\ +(K - N_{\mathtt{H}_1})[N_{\mathtt{H}_1}, X_{\mathtt{H}_1}]) = \widetilde{a^{+}_{\mathtt{H}_1}}\widetilde{a^{-}_{\mathtt{H}_1}} + \widetilde{a^{-}_{\mathtt{H}_1}}\widetilde{a^{+}_{\mathtt{H}_1}}.\end{gathered} \tag{7.19}$$

Using (7.10),(7.11) we also have:

$$\begin{gathered}\widetilde{H}_{\mathtt{H}_1} = (X_{\mathtt{H}_1} - 2N_{\mathtt{H}_1})^2 - 2(\alpha + 1)(X_{\mathtt{H}_1} - 2N_{\mathtt{H}_1}) \\ -(2K - X_{\mathtt{H}_1})^2 - 2(\alpha + 1)(2K - X_{\mathtt{H}_1}) = 4({N_{\mathtt{H}_1}}^2 - K^2) \\ +2((K - N_{\mathtt{H}_1})X_{\mathtt{H}_1} + X_{\mathtt{H}_1}(K - N_{\mathtt{H}_1})) + 4(\alpha + 1)(N_{\mathtt{H}_1} - K).\end{gathered} \tag{7.20}$$

Moreover, the energy levels are

$$\lambda_n = 4(n^2 + (\alpha + 1)n + \frac{\alpha + 1}{2}), \qquad n \geq 0. \tag{7.21}$$

The following theorem is valid.

Theorem 7.1. *The equation*

$$\widetilde{H}_{\mathtt{H}_1}\psi_n(x) = \lambda_n\psi_n(x), \qquad n \geq 0, \tag{7.22}$$

where λ_n is defined by (7.21), is equivalent to the differential equation for the Laguerre polynomials:

$$x(L_n^{(\alpha)}(x))'' + (\alpha + 1 - x)(L_n^{(\alpha)}(x))' + nLn^{(\alpha)}(x) = 0, \quad n \geq 0. \tag{7.23}$$

Proof. Using (7.9) and (7.4) we rewrite the differential equation (7.23) in the form of the operator equality in the space $\mathtt{H}_1$:

$$K^2 + \alpha K - X_{\mathtt{H}_1}(K - N_{\mathtt{H}_1}) = 0. \tag{7.24}$$

In view of (7.16) and (7.17) the equation (7.22) is equivalent to the following operator equality in the space $\mathtt{H}_1$:

$$K^2 + \alpha K - X_{\mathtt{H}_1}(K - N_{\mathtt{H}_1}) = \frac{1}{2}[K - N_{\mathtt{H}_1}, X_{\mathtt{H}_1}] - K - \frac{\alpha + 1}{2}I_{\mathtt{H}_1}. \tag{7.25}$$

It is obvious from (7.24) and (7.25) that it is sufficient to prove that the right-hand side of (7.25) vanishes.From (7.16) it follows that the latter is true. ∎

Remark 7.2. The theorem 2.23 is true in our case. Then we have

$$[\widetilde{a_{\mathtt{H}_1}^-}, \widetilde{a_{\mathtt{H}_1}^+}]\psi_n(x) = 2({b_n}^2 - {b_{n-1}}^2)\psi_n(x), \quad n \geq 0, \quad b_{-1} = 0. \tag{7.26}$$

Taking into account (7.8) we calculate

$${b_n}^2 - {b_{n-1}}^2 = 2n + \alpha + 1. \tag{7.27}$$

From (7.27) and (7.26) we get the following commutation relation:

$$[\widetilde{a_{\mathtt{H}_1}^-}, \widetilde{a_{\mathtt{H}_1}^+}] = 2N_{\mathtt{H}_1} + (\alpha+1)I_{\mathtt{H}_1}. \tag{7.28}$$

As another instance of the nonsymmetric scheme we consider the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ under the condition $\alpha \neq \beta$.

8. The Jacobi polynomials

The Jacobi polynomials ([14]) one can be determined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \tag{8.1}$$

Let

$$\mathtt{G}_1 = L^2([-1,1]),$$
$$\mathtt{H}_1 = L^2([-1,1]; (d_0(\alpha,\beta))^{-2}(1-x)^\alpha(1+x)^\beta dx),$$

where

$$d_0^2(\alpha,\beta) = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Let

$$\psi(x) = (1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}}. \tag{8.2}$$

We define the orthonormal systems $\{\psi_n(x)\}_{n=0}^\infty$ and $\{\phi_n(x)\}_{n=0}^\infty$ by the following formulas:

$$\psi_n(x) = d_0 d_n^{-1} P_n^{(\alpha,\beta)}(x), \qquad \phi_n(x) = \psi(x)\psi_n(x), \qquad n \geq 0, \tag{8.3}$$

where the constants d_n are given by

$$d_n^2 = \frac{2^{\alpha+\beta+1}(\Gamma(n+\alpha+1))(\Gamma(n+\beta+1))}{(2n+\alpha\beta+1)n!\Gamma(n+\alpha+\beta+1)}, \quad n \geq 0. \tag{8.4}$$

Using (8.4) and the recurrence relations for the Jacobi polynomials (see [14]) we get (6.4), (6.5), where

$$a_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \quad n \geq 0, \tag{8.5}$$

$$b_n = \sqrt{\frac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}}, \quad n \geq 0. \tag{8.6}$$

It is known how the operator A defined by (4.5) acts on the Jacobi polynomials ([14]). Then it is not hard to get from (8.3),(8.4) the following equalities:

$$A\psi_n(x) = (n+\alpha+\beta+1)b_{n-1}\psi_{n-1}(x) -$$
$$\frac{2n(n+\alpha+\beta+1)}{\alpha+\beta}a_n\psi_{n-1}(x) - nb_n\psi_{n+1}(x), \qquad n \geq 0, \quad b_{-1} = 0. \tag{8.7}$$

Multiplying both sides of (6.4) by $n+\alpha+\beta+1$ and subtracting (8.7) from the obtained result we eliminate ψ_{n-1} from (8.7). Then we obtain

$$\frac{\sqrt{2}}{2n+\alpha+\beta+1}\Big(x(n+\alpha+\beta+1)-A-\frac{a_n(n+\alpha+\beta+1)(2n+\alpha+\beta)}{\alpha+\beta}\Big)$$
$$\psi_n(x)=\sqrt{2}b_n\psi_{n+1}(x), \qquad n\geq 0. \tag{8.8}$$

From (8.8),(6.17) and (2.36) it follows that

$$\begin{aligned}\widetilde{a^+_{\mathtt{H}_1}} &= \sqrt{2}((X_{\mathtt{H}_1}(N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})-A)\\ &(2N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})^{-1}+(\alpha-\beta)(N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})\\ &(2N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})^{-1}(2N_{\mathtt{H}_1}+(\alpha+\beta+2)I_{\mathtt{H}_1})^{-1}).\end{aligned} \tag{8.9}$$

In order to eliminate ψ_{n+1} from (8.7) we multiply both sides of (6.4) by n and add this to (8.7). Then we get

$$\frac{\sqrt{2}}{2n+\alpha+\beta+1}\Big(xn+A+\frac{a_n n(2n+\alpha+\beta+2)}{\alpha+\beta}\Big)$$
$$\psi_n(x)=\sqrt{2}b_{n-1}\psi_{n-1}(x), \qquad n\geq 0. \tag{8.10}$$

Combining (8.10),(6.17) and (2.36) we have

$$\begin{aligned}\widetilde{a^-_{\mathtt{H}_1}} &= \sqrt{2}(X_{\mathtt{H}_1}N_{\mathtt{H}_1}+A-(\alpha-\beta)N_{\mathtt{H}_1}\\ &(2N_{\mathtt{H}_1}+(\alpha+\beta)I_{\mathtt{H}_1})^{-1})(2N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})^{-1}.\end{aligned} \tag{8.11}$$

Taking into account (6.17) and (8.9),(8.11) one can write

$$\begin{aligned}\widetilde{X}_{\mathtt{H}_1} &= \frac{1}{\sqrt{2}}(\widetilde{a^+_{\mathtt{H}_1}}+\widetilde{a^-_{\mathtt{H}_1}}) = X_{\mathtt{H}_1}N_{\mathtt{H}_1}+\\ &(\alpha-\beta)(2N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})^{-1}((2\alpha+2\beta+1)N_{\mathtt{H}_1}+(\alpha+\beta+1)^2)\\ &(2N_{\mathtt{H}_1}+(\alpha+\beta)I_{\mathtt{H}_1})^{-1})(2N_{\mathtt{H}_1}+(\alpha+\beta+2)I_{\mathtt{H}_1})^{-1}),\end{aligned} \tag{8.12}$$

$$\begin{aligned}\widetilde{P}_{\mathtt{H}_1} &= \frac{-\imath}{\sqrt{2}}(\widetilde{a^+_{\mathtt{H}_1}}-\widetilde{a^-_{\mathtt{H}_1}}) = -\imath(-2A+X_{\mathtt{H}_1}(\alpha+\beta+1))+\\ &(\alpha-\beta)((2N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})^2+N_{\mathtt{H}_1})(2N_{\mathtt{H}_1}+(\alpha+\beta)I_{\mathtt{H}_1})^{-1}\\ &(2N_{\mathtt{H}_1}+(\alpha+\beta+2)I_{\mathtt{H}_1})^{-1})(2N_{\mathtt{H}_1}+(\alpha+\beta+1)I_{\mathtt{H}_1})^{-1}.\end{aligned} \tag{8.13}$$

It is not hard to find from (8.12),(8.13) and the definition (6.15) the explicit form of the hamiltonian $\widetilde{H}_{\mathtt{H}_1}$. In view of (6.16),(8.6) we get the energy levels by

$$\begin{aligned}\lambda_n &= 2(b_{n-1}^2+b_n^2)=\\ &=\frac{(2n+\alpha+\beta+1)^2(s_n-4w_n)+5s_n-2w_n}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+2)^2(2n+\alpha+\beta-1)(2n+\alpha+\beta+3)},\end{aligned} \tag{8.14}$$

where

$$\begin{aligned}s_n &= t_n+t_{n+1}, \qquad t_n=n(n+\alpha)(n+\beta)(n+\alpha+\beta),\\ w_n &= 2n^2+2n(\alpha+\beta+1)+(\alpha+1)(\beta+1).\end{aligned} \tag{8.15}$$

The next theorem is an extension of the theorem 4.4.

Theorem 8.1. *The equation* $\widetilde{H}_{\mathtt{H}_1}\psi_n(x) = \lambda_n\psi_n(x), \quad n \geq 0$, *where* λ_n *defined by (8.14) is equivalent to the differential equation for the Jacobi polynomials:*

$$\frac{d}{dx}((1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx})P_n^{(\alpha,\beta)}(x) + \\ +n(n+\alpha+\beta+1)(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = 0, \tag{8.16}$$

$(n \geq 0.)$

Remark 8.2. The result may be proved in much the same way as the theorem 4.4. Here we omit this proof as well as the explicit forms for the number operator $\widetilde{N}_{\mathtt{H}_1}$, the momentum operator $\widetilde{P}_{\mathtt{H}_1}$ and the hamiltonian $\widetilde{H}_{\mathtt{H}_1}$.

In conclusion we will point out some associations between the canonical systems in the symmetric and nonsymmetric schemes.

9. Connection of symmetric with nonsymmetric schemes

Denote by μ^s - a symmetric probability measure on R^1 and by $\mathtt{H}_1^s = L^2(R^1;\mu^s)$. Let $\mu^s(0) = 0$ and

$$\mu^s = \mu_+ + \mu_- \tag{9.1}$$

be a decomposition of the measure μ^s into the orthogonal sum of two (nonsymmetric) measures μ_+ and μ_- defined by the equalities:

$$\mu_+(B) = \mu^s(R^1_+ \cap B), \qquad \mu_-(B) = \mu^s(R^1_- \cap B), \tag{9.2}$$

for any Borel set $B \subset R^1$. Let $\{\psi_n(x)\}_{n=0}^\infty$ be a canonical (complete) orthonormal polynomial system in the space $\mathtt{H}_1^s$. Suppose that this system is constructed by the sequence $\{b_n^s\}_{n=0}^\infty$ via the recurrence relations (2.5) ,(2.6). Further, we denote by

$$\mathtt{H}_1^+ = L^2(R^1_+; 2\mu_+(dx)), \quad \mathtt{H}_1^- = L^2(R^1_-; 2\mu_-(dx)). \tag{9.3}$$

Let $\{\psi_n^+(x)\}_{n=0}^\infty$, $\{\psi_n^-(x)\}_{n=0}^\infty$ be canonical (complete) orthonormal polynomial systems in $\mathtt{H}_1^+$ and $\mathtt{H}_1^-$ respectively. Let these systems be constructed by real sequences $\{a_n^a\}_{n=0}^\infty$, $\{b_n^a\}_{n=0}^\infty$ via the recurrence relations:

$$x\psi_n^+(x) = b_n^a\psi_{n+1}^+(x) + a_n^a\psi_n^+(x) + b_{n-1}^a\psi_{n-1}^+(x), \quad n \geq 0, \quad b_{-1}^a = 0, \tag{9.4}$$

$$\psi_0^+(x) = 1, \tag{9.5}$$

and

$$x\psi_n^-(x) = b_n^a\psi_{n+1}^-(x) - a_n^a\psi_n^-(x) + b_{n-1}^a\psi_{n-1}^-(x), \quad n \geq 0, \quad b_{-1}^a = 0, \tag{9.6}$$

$$\psi_0^-(x) = 1. \tag{9.7}$$

The following lemmas are valid. The proof of these is left to the reader.

Lemma 9.1. *Let* $2\mu_+$ *and* $2\mu_-$ *be probability measures on* R^1_+ *and* R^1_- *respectively such that the measure* $\mu^s = \mu_+ + \mu_-$ *is a symmetric probability measure on* R^1. *Let* $\{\psi_n^+(x)\}_{n=0}^\infty$ *and* $\{\psi_n^-(x)\}_{n=0}^\infty$ *be the orthonormal polynomial systems constructed by* $2\mu_+$ *and* $2\mu_-$ *respectively via the recurrence relations (9.4) -(9.7). Then the polynomial system* $\{\psi_n(x)\}_{n=0}^\infty$ *defined by:*

$$\psi_n(x) = \frac{1}{2}(\psi_n^+(x) + \psi_n^-(x)), \quad n \geq 0, \tag{9.8}$$

is an orthonormal system in the space

$$\mathtt{H}_1^s = \mathtt{H}_1^+ \oplus \mathtt{H}_1^-, \tag{9.9}$$

where the measure μ^s is determined by (9.1) and satisfies the recurrence relations (2.5) ,(2.6) with the coefficients $\{b_n^a\}_{n=0}^{\infty}$.

Lemma 9.2. *Let $\{\psi_n(x)\}_{n=0}^{\infty}$ be a canonical orthonormal polynomial system in $\mathtt{H}_1^s$, constructed by a symmetric probability measure μ^s (for more details we refer the reader to the section 2) via the recurrence relations (2.5) ,(2.6). Then the polynomial systems $\{\psi_{2l}(x)\}_{l=0}^{\infty}$ and $\{\psi_{2l+1}(x)\}_{l=0}^{\infty}$ are the orthonormal (it is understood that they are incomplete) systems in the spaces $\mathtt{H}_1^+$ and $\mathtt{H}_1^-$ respectively.*

Remark 9.3. 1.One can obtain the corresponding complete orthonormal systems $\{\psi_n^+(x)\}_{n=0}^{\infty}$ in $\mathtt{H}_1^+$ and $\{\psi_n^-(x)\}_{n=0}^{\infty}$ in $\mathtt{H}_1^-$ from the set $\{\psi_n(x)\}_{n=0}^{\infty}$ by the Schmidt orthogonalization in $\mathtt{H}_1^+$ and $\mathtt{H}_1^-$ respectively.

2. There is a simple relation between the moments μ_k^s and μ_k^+, μ_k^-; however connections between corresponding recurrence relations as well as between corresponding oscillators are rather complicated.

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